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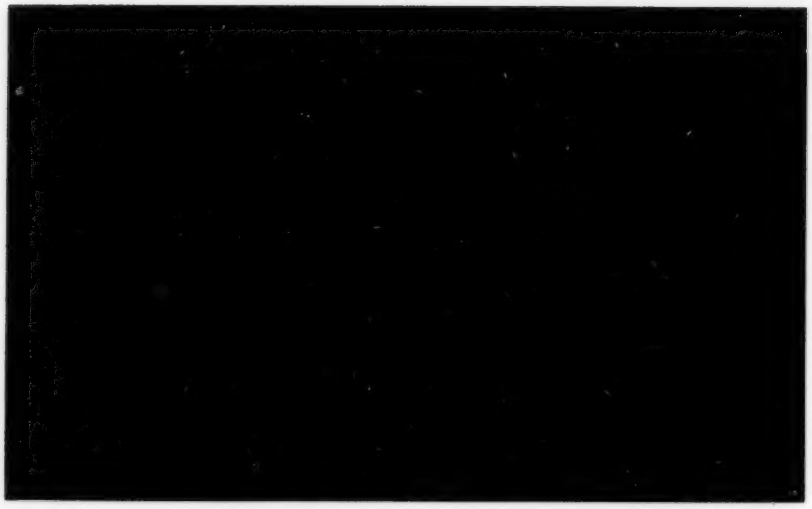

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On Descent Simplicial Decomposition Methods for the Monotone Variational Inequality Problem with its Application to the Network Equilibrium Problem

by

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RÉSUMÉ

Méthodes de décomposition simpliciale de descente pour le problème d'inégalité variationnelle monotone avec une application sur le problème d'équilibre de réseaux.

D'abord, nous développons une famille de méthodes de descente pour le problème d'inégalité variationnelle monotone et donnons une preuve de convergence où l'hypothèse de monotonicité forte est remplacée par l'hypothèse de monotonicité seulement, basée sur une reformulatiom du problème d'inégalité variationnelle monotone. Ensuite, ces méthodes de descente sont incorporées à une méthode de décomposition simpliciale pour résoudre le problème où une séquence de problèmes d'inégalité variationnelle monotone affine symétrique aussi bien qu'asymétrique sont résolus. De par la structure particulière des contraintes du problème d'inégalité variationnelle restreint dans la méthode de décomposition simplicielle, les deux problèmes peuvent être résolus plus facilement, ce qui facilite également l'évaluation de la fonction objective du problème d'optimisation associé au problème d'inégalité variationnelle restreint. Ces méthodes sont adaptées au problème d'équilibre de réseaux et des résultats numériques pour un petit réseau sont présentés.

Mots-clés : inégalités variationnelles, problème d'équilibre de réseau, méthodes de décomposition simpliciale, méthodes de descente.

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ABSTRACT

In this paper, we first develop a family of descent methods for a monotone variational inequality problem and provide a proof of their convergence where the strong monotonicity assumption is replaced by the monotonicity assumption based on a reformulation of the variational inequality problem. Then these descent methods are incorporated into a simplicial decomposition method to solve the variational inequality problem where a series of both symmetric and assymetric affine variational inequality problems are solved. Due to a special constraint structure (a unit simplex) of a restricted variational inequality problems in the simplicial decomposition method, both methods can be solved more easily, which also facilitates the evaluation of the objective function of the optimization formulation of the restricted variational inequality problem. The resulting methods are adapted for the network equilibrium problem and the computational results for a small network are reported.

Key words : Network equilibrium problem, Variational Inequalities, Simplicial decomposition, Descent method.

1 Introduction

Let Ω be a nonempty and closed convex set of R^n and s be a continuous mapping from R^n into R^n. In this paper, we consider the variational inequality problem that consists in finding a vector $v^* \in \Omega$ such that

$$s(v^*)^T(v - v^*) \geq 0, \forall v \in \Omega. \tag{1}$$

It is well known that when $\nabla s(v)$ is symmetric and positive definite on Ω, the variational inequality problem corresponds to the first order optimality condition of a convex program; the reverse is not necessarily true. The variational inequality problem has been widely used to model network equilibrium problems arising in transportation system analysis (Smith, 1979, 1983, Dafermos, 1980, Bertsekas and Gafni,1982, Florian and Spiess, 1984, Florian, 1986, Marcotte and Guélat, 1988) and regional science (Florian and Los, 1982, Nagurney, 1987, Dafermos and Nagurney, 1987). The study of variational inequality problems draws on nonlinear optimization methodology, solution of nonlinear equations and fixed point theory.

During the past several decades, there have been many solution procedures developed for various purposes depending on the special structures of the problems on hand. In general there are three basic classes of the algorithms. The first are the function-related methods. These methods are proposed based on the idea of approximating the function s linearly or nonlinearly (Pang and Chan, 1982, Cohen, 1986) or perturbating the function s (Rockafellar, 1976); The Cohen's method has been discussed on its relationship with other methods (Wu, 1991). The second are the region-related methods such as inner approximation like the simplicial decomposition methods (Hearn et al., 1986, Pang and Yu, 1986) and outer approximation such as that by Fukushima (1986) of the feasible region; finally the third are the optimization-reformulation based methods where one reformulate the asymmetric variational inequality problem as an optimization problems like Neguyen and Dupuis (1983), Smith (1983), Marcotte (1986), Marcotte and Dussault (1989), Fukushima

(1989), Taji et al (1991). These methods later are summarized and extended in a general descent framework developed by Wu et al. (1991) and Wu (1991) where several descent methods were further investigated and the strong monotonicity is assumed and the compactness on the feasible region is relaxed. For a literature review, see a recent paper by Harker and Pang (1990). Recently Zhu et al (1992) developed a very general descent framework which includes the newton method still under the strong monotonicity assumption of $s(v)$. Computational experiences (to be reported later) show that the descent methods are in general very robust for convergence as compared with the some iterative methods (i.e., with a unit step size) in the function-related methods. In this case, however, it has to solve an affine variational inequality problem on the feasible set Ω for finding a search direction and an evaluation of the cost function is equivalent to a solution of a symmetric affine variational inequality problem. The solutions of affine variational inequality problems are very expensive in general unless Ω has a simple structure. In the network equilibrium problem, Ω may not be bounded. Based on the these observations, our approach combines various advantages of the descent methods with the simple constraint structure of a restricted variational inequality problem by incorporating the simplicial decomposition method for the variational inequality problem. More specifically, our basic characteristics of the research are the following:

- a family of descent methods including the descent Newton method are developed to solve only monotone (not strongly) variational inequality problem with a proof of their convergence on a compact Ω;
- while the methods can be so used, we adopt a simplicial decomposition method; thus the feasible region for the restricted variational inequality problem is a unit-simplex which is convex and compact;
- then the descent methods are used for solving the restricted variational inequality problem on the unit-simplex;

- it is easier to solve the restricted variational inequality problem on the unit-simplex than variational inequality problem on Ω by using the methods like the Lemke's pivoting method (Eaves, 1978);
- due to the similar reason, the evaluation of the objective becomes simpler.

Though the basic idea of the approach seems a very natural extension of the existing theory, there is few convergence proofs of these descent methods if the monotonicity of $s(v)$ is assumed and few computational experience of these methods. We mention that recently Zhu et al. (1992) proposed a family of descent methods where the strong monotonicity and the differentiability assumptions on $s(v)$ are relaxed but the Newton iteration is not included. The obvious difficulty is that there is no guarantee on the strict descent direction at each iteration. It is our intention to show that this difficulty can be removed by adjusting some parameters in the reformulation of the variational inequality problem and also see how the approach works on the network equilibrium problem both theoretically and numerically.

This paper is organized as follows. In section 2 we give some basic definitions and results in variational inequality problem. Section 3 is devoted to develop a family of the descent methods for the variational inequality problem and provide corresponding proofs. In section 4 we consider the descent simplicial decomposition method and discuss the various steps. In section 5, we first discuss the network equilibrium problem and its adaptation of the descent simplicial decomposition method and report computational results for a small network equilibrium problem. Finally, in section 6, we conclude the paper and provide the future direction of research.

2 Preliminaries and results

In this section, we give some definitions and assumptions that are used in our analysis.

The mapping $s(v) : R^n \rightarrow R^n$ is said to be monotone on Ω if

$$(s(v') - s(v''))^T (v' - v'') \geq 0, \forall v', v'' \in \Omega, \tag{2}$$

strictly monotone on Ω if the strict inequality holds in (2) for $v' \neq v''$ and strongly monotone on Ω if there is a positive b such that

$$(s(v') - s(v''))^T(v' - v'') \geq b||v' - v''||_2^2, \forall v', v'' \in \Omega \tag{3}$$

It is well known (Ortega and Rheinboldt, 1970) that, if $s(v)$ is continuously differentiable and the Jacobian matrix $\nabla s(v)$ is positive definite on Ω,

$$(v' - v'')^T \nabla s(v)(v' - v'') \geq b||v' - v''||_2^2, \forall\ v', v'' \in \Omega, \quad v' \neq v'' \tag{4}$$

then s is strongly monotone on Ω with b the minimum eigenvalue of the symmetric part of $\nabla s(v)$ over Ω, which we denote b_{min}. Note that $\nabla s(v)$ need not be symmetric in (4).

If the mapping $s(v) : R^n \rightarrow R^n$ is Lipschitz continuous on Ω, then there is a constant L_s such that

$$||s(v') - s(v'')||_2 \leq L_s||v' - v''||_2, \forall\ \ v', v'' \in \Omega. \tag{5}$$

Let D be any $n \times n$ symmetric positive definite matrix. The projection under the D-norm of a point $v - D^{-1}s(v)$ onto the set Ω, denoted as $P(v)$, is defined as the solution (which must exist and be unique) to the following mathematical program:

$$\min_{u \in \Omega} ||u - (v - D^{-1}s(v))||_D^2$$

where $||x||_D = (x^T D x)^{\frac{1}{2}}$ denotes the D-norm of a vector $x \in R^n$ which is equivalent to the symmetric affine variational inequality problem, i.e., find $u^* = P(v)$ such that:

$$[s(v) + D(u^* - v)]^T(u - u^*) \geq 0, \forall u \in \Omega.$$

We note that D can be a variable matrix denoted as $D(v)$ which is scaled to be $\frac{D(v)}{\alpha_1}$ with α_1 is a positive parameter. For an any $u \in \Omega$, we consider the following affine variational inequality problem, i.e., find $u^* \in \Omega$, such that

$$[s(v) + \frac{1}{\alpha} B(v)^T(u^* - v)]^T(u - u^*) \geq 0, \forall u \in \Omega, \tag{6}$$

where $B(v)$ is positive semi-definite matrix and α is a positive parameter. We denote the solution set of (6) as $\Gamma(v)$ and let $H(v) \in \Gamma(v)$. It is well known that if $H(v^*) = v^*$, then v^* is the solution of the variational inequality problem (1).

In the following, we discuss some important results for the monotone variational inequality problem. Consider the following equivalent mathematical programming for the variational inequality problem which is first proved by Fukushima (1989) and further considered by Wu et al. (1991):

$$\min_{v \in \Omega} f(v) = -s(v)^T(P(v) - v) - \frac{1}{2\alpha_1}(P(v) - v)^T D(v)(P(v) - v) \tag{7}$$

$$\text{s.t.} \quad P(v) = \arg\min_{u \in \Omega} s(v)^T(u - v) + \frac{1}{2\alpha_1}(u - v)^T D(v)(u - v) \tag{8}$$

where $D(v)$ is a symmetric and positive definite matrix and α_1 is a positive parameter.

Proposition 2.1 *Assume that $s(v)$ is continuously differentiable. Let I_n, b and α_1 be defined as before and λ_D be the minimum eigenvalue of $D(v)$.*

P1) *$f(v)$ is Lipschitz continuous on Ω;*

P2) $\nabla f(v) = s(v) - (\nabla s(v) - \frac{D(v)}{\alpha_1})(P(v) - v) - \frac{1}{2\alpha_1}(P(v) - v)^T \nabla D(v)(P(v) - v)$.

Proof. See Proposition 3.3 and Proposition 3.5 in Wu et al (1991) where $\phi(u, v)$ is defined as

$$\phi(u, v) = \frac{1}{2\alpha_1}(u - v)^T D(v)(u - v)$$

for the proof of P1) and P2). □

Proposition 2.2 *Suppose that $D(v)$ is a symmetric positive definite matrix. $f(v) \geq 0$ for all $v \in \Omega$ and $f(v^*) = 0$, i.e., v^* solves the optimization problem (7) and (8) if and only if v^* solves the variational inequality problem.*

Proof. See Fukushima (1989) or Wu et al (1991).

Remark. There is no requirement for the monotonicity assumption on $s(v)$ in the above proposition.

3 A globally convergent method

In the following, we develop a family of modified descent methods to solve the optimization problem (7) and (8). The following proposition provides us an important information as to either $H(v) - v$ is a strictly descent direction of $f(v)$ at v or $f(v)$ is bounded above.

Proposition 3.1 *Suppose that $D(v)$ is differentiable, s is a continuously differentiable function and α_1 and α are both positive parameters. Suppose that we have either*

A1) $\frac{B(v)}{\alpha} = \nabla s(v)$,

or

A2) $H(v) = P(v)$.

If v is not a solution of the variational inequality problem, then either $d = H(v) - v$ is the descent direction of $f(v)$ at v or $f(v) \leq \frac{\pi_{max}}{\alpha_1}$ where π_{max} is a bounded value.

Proof.

$$\begin{aligned}
\nabla f(v)^T(H(v)-v) &= [s(v) - (\nabla s(v) - \frac{D(v)}{\alpha_1})(P(v)-v)]^T(H(v)-v) \\
&\quad -\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v) \\
&= s(v)^T(H(v)-v) - (P(v)-v)^T(\nabla s - \frac{D(v)}{\alpha_1})^T(H(v)-v) \\
&\quad -\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v) \\
&= s(v)^T(H(v)-v) - (P(v)-v)^T[s(v) + \nabla s(v)(H(v)-v)] \\
&\quad +(P(v)-v)^T s(v) + (P(v)-v)^T\frac{D(v)}{\alpha_1}(H(v)-v) \\
&\quad -\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v) \\
&= s(v)^T(H(v)-v) - (H(v)-v)^T[s(v) + \nabla s(v)(H(v)-v)]
\end{aligned}$$

$$
\begin{aligned}
& -(P(v)-H(v))^T[s(v)+\nabla s(v)(H(v)-v)] \\
& +(P(v)-v)^T s(v)+(P(v)-v)^T\frac{D(v)}{\alpha_1}(H(v)-v) \\
& -\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v) \\
= \; & -(H(v)-v)^T\nabla s(v)^T(H(v)-v) \\
& -(P(v)-H(v))^T[s(v)+\frac{B(v)^T}{\alpha}(H(v)-v)] \\
& -(P(v)-H(v))^T(\nabla s(v)-\frac{B(v)}{\alpha})^T(H(v)-v) \\
& +s(v)^T(P(v)-v)+(P(v)-v)^T\frac{D(v)}{\alpha_1}(H(v)-v) \\
& -\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v) \\
= \; & T_1+T_2+T_3+T_4+T_5.
\end{aligned}
$$

According to the definition of the monotonicity of $s(v)$ and $H(v)$, $T_1 \le 0$, $T_2 \le 0$ and $T_3 = 0$ due to A1) or A2). Now we consider the term T_4.

$$
\begin{aligned}
T_4 &= s(v)^T(P(v)-v)+\frac{1}{\alpha_1}(P(v)-v)^T D(v)(H(v)-v) \\
&= s(v)^T(P(v)-v)+\frac{1}{2\alpha_1}(P(v)-v)^T D(v)(P(v)-v) \\
&\quad +\frac{1}{2\alpha_1}(P(v)-v)^T D(v)(P(v)-v)+\frac{1}{\alpha_1}(P(v)-v)^T D(v)(H(v)-P(v)) \\
&= -f(v)+\frac{1}{2\alpha_1}(P(v)-v)^T D(v)(P(v)-v)+\frac{1}{\alpha_1}(P(v)-v)^T D(v)(H(v)-P(v)).
\end{aligned}
$$

There are two exclusive cases.

Case 1 where $T_4 + T_5 < 0$. Then we have

$$\nabla f(v)^T(H(v)-v) < 0.$$

Case 2 where $T_4 + T_5 \ge 0$. Then we obtain

$$
f(v) \le \frac{1}{2\alpha_1}(P(v)-v)^T D(v)(P(v)-v)+\frac{1}{\alpha_1}(P(v)-v)^T D(v)(H(v)-P(v))+
$$

$$\frac{1}{2\alpha_1}(P(v)-v)^T\nabla D(v)(P(v)-v)(H(v)-v).$$

From the compactness (i.e., also bounded) of Ω and the differentiability of $D(v)$, there exists such a positive parameter π_{max} that

$$f(v) \leq \frac{\pi_{max}}{\alpha_1}.$$

We obtain the desired result. □

Remark.

- If the two positive parameters α_1 and α are equal and $D(v)$ and $B(v)$ are equal and symmetric positive definite, then A2 is true.
- The two cases can be easily identified by determining the stepsize t^*, i.e.,

$$t^* = \arg\min_{0\leq t\leq 1}\{f(v+t(H(v)-v)\} \tag{9}$$

If $t^* > 0$, then we have the case 1 and case 2 otherwise.

Based on the study, we first state the algorithm **VIP** for the variational inequality problem and then the convergence proposition.

VIP

Initialization	1	Let $v^0 \in \Omega, l = 0, \beta > 1$, α_1 and α are positive parameters
Iteration	2	If convergence tests are satisfied, then stop;
	3	Solve the affine variational inequality problem: $II(v^l)$ in (6);
	4	Calculate the step size t^* in (9):
	5	Identify the two cases:
		If $t^* > 0$, then $v^{l+1} = v^l + t^*(H(v^l) - v^l)$;
		otherwise $\alpha_1^{l+1} = \beta\alpha_1^l$ and $v^{l+1} = v^l$;

End Iteration

Proposition 3.2 (Global Convergence) *Suppose that the mapping s is continuously differentiable monotone on a nonempty, convex and compact region Ω. Let either $\frac{B(v)}{\alpha} = \nabla s(v)$ or $P(v) = H(v)$. If α_1^l is selected according to the* **VIP**, *then for any feasible solution $v^0 \in \Omega$, the sequence $\{v^l\}$ generated by the algorithm* **VIP** *is lies in Ω and converges to a solution of the variational inequality problem.*

Proof. Since the points v^l and $H(v^l)$ both belong to Ω and since $0 \leq t \leq 1$, it follows from the convexity of Ω that the sequence $\{v^l\}$ lies in Ω. If the **VIP** stops, then some convergence tests are satisfied and we have a solution of the variational inequality problem. Now let us consider an infinite sequence generated by the **VIP**. We have following two exclusive subsequences.

Subsequence 1 where $\alpha_1^{l'} \to \infty$.

In this case, we have

$$f(v^l) \leq \frac{\pi_{max}}{\alpha_1^l}.$$

Since $P(v)$ is continuous in α_1 and v, so is $f(v)$ (Auslender, 1976, Theorem 1.7, Chapter 4). From the compactness of Ω, we have $f(v^l) = f(\hat{v}) = 0$ as $\alpha_1^l \to \infty$ where $\hat{v}$ is the limit if the subsequence.

According to Proposition 2.2, $\hat{v}$ is a solution of the variational inequality problem.

Subsequence 2 where there exists an index l_{min} such that $\alpha_1^{l'} = \alpha_1^{l_{min}}, \forall\ l' \geq l_{min}$. Thus $\forall\ l' \geq l_{min}$,

$$\nabla f(v^{l'})^T (H(v^{l'}) - v^{l'}) < 0.$$

Since the Ω is compact and the algorithmic map is upper semicontinuous due to the fact that $H(v) - v$ is upper semicontinuous (see a similar proof of the assertion in Wu et al (1991)), we have $f(v^{l'}) \rightarrow 0$. This implies by Proposition 2.2 that $\{v^{l'}\}$ is convergent to a solution of the variational inequality problem by Zangwill's Convergence Theorem A (Zangwill, 1969). Therefore there exists a subsequence which is convergent to a solution of the variational inequality problem. □

Remark.

- If $B(v) = \nabla s(v)$ and $\alpha = \alpha_1 = 1$, we obtain the descent Newton algorithm of Taji et al (1991). However we have considerably relaxed the condition imposed there where $b - \frac{\|D\|}{2\alpha_1} > 0$ is required.

- If $D(v) = B(v)$ and $\alpha_1 = \alpha$, we have $H(v) = P(v)$ and therefore arrive at the general framework (Wu et al, 1991) where the strong monotone assumption on $s(v)$ is required and α_1 is fixed. Recently, Zhu et al (1992) also obtain the similar algorithm where $P(v) = H(v)$ and $\alpha_1 = \alpha$ are assumed only.

- In the algorithm, the solution of both symmetric and asymmetric affine variational inequality problems can be solved by appling Lemke's complementary pivoting method (Eaves, 1978) or more recently a matrix splitting method (Pang, 1991). These methods, however, do not take advantage of special structures of Ω which is the unit simplex as example.

- The stepsize is conducted by using Amijro rule (Ortega and Rheinboldt, 1970).

- The convergence tests can be done according to $f(v)$ and $H(v) - v$. If $f(v)$ or $\|H(v) - v\|_2$ is close to zero, then stop.

- In terms of the selection of α and α_1, we have the following two cases. First, if $\frac{B(v)}{\alpha} = \nabla s(v)$, then there exists a positive parameter α_1^{min} such that $\alpha_1^{min} \leq \alpha_1$ according to the Proposition 3.1. Second, if $P(v) = H(v)$, i.e., $\alpha_1 = \alpha$, then there exists a positive parameter α_1^{min} such that $\alpha_1^{min} \leq \alpha_1$ for the same reason.

4 The descent simplicial decomposition method

In this section, we assume that Ω is a polyhedron and present a descent simplicial decomposition method (DSDM) and discuss steps of the method. The basic idea of the method is similar to other simplicial decomposition methods (Lawphongpanich and Hearn, 1984, 1986, Pang and Yu, 1988) except that each restricted variational inequality problem is solved by using the **VIP** discussed in the previous section instead of the function-related methods where in fact the stepsize is fixed to be one (Pang and Chan, 1982). Main point here is that any monotone variational inequality problem with a polyhedron can be solved with a series of restricted variational inequality problems with a unit simplex. No matter what methods are used, a series of affine variational inequality problems are required to be solved. Now we give the following descent simplicial decomposition method (**DSDM**) without proof.

DSDM

Initialization	1	Find two extreme points $e^1, e^2 \in \Omega$. Let $\hat{G}^2 = \infty, G^2 = \infty, \beta$ be a positive parameter and $l = 2$. Let Ω^l be the convex hull of the columns of E^l where E^l is a collection of extreme points found at iteration l and ϵ^l be an error such that $\epsilon^l \to 0$ as $l \to \infty$. And let $E^2 = \{e^1, e^2\}$.
Iteration	2	If some convergence tests are satisfied, stop;
	3	Solve the restricted variational inequality problem, i.e., find $v^{l+1} \in \Omega^l$ such that $s(v^{l+1})^T(v - v^{l+1}) \geq -\epsilon^l, \forall v \in \Omega^l$, which is equivalent to find a vector of $\lambda^* \in \Omega^l_\lambda = \{\lambda \mid \sum_{n=1}^{l} \lambda_n = 1, \lambda \geq 0\}$ such that $s(E^l\lambda^*)^T E^l(\lambda - \lambda^*) \geq 0, \forall \lambda \in \Omega^l_\lambda$.
	4	Solve the linear program problem, i.e., $e^* = \arg\min_{e \in \Omega} s(v^{l+1})e$;
	5	Update the gap function, i.e., $G^{l+1} = -s(v^{l+1})(e^* - v^{l+1})$;
	6	Generate E^{l+1}, i.e., If $G^{l+1} \leq \hat{G}^l - \beta$, then $E^{l+1} = E^l - D^l \bigcup \{e^*\}$; otherwise, $E^{l+1} = E^l \bigcup \{e^*\}$; where D^l = columns of E^l with zero weight in the expression of v^{l+1} as convex combination of columns of E^l. Let $\hat{G}^{l+1} = \min\{\hat{G}^l, G^{l+1}\}, l = l + 1$.
End Iteration		

In step 1, e^1 and e^2 can be found by solving twice linear programs. In step 3, the restricted

variational inequality problem is to find $\lambda^* \in \Omega_\lambda^l$ such that

$$\hat{s}(\lambda^*)^T(\lambda - \lambda^*) \geq 0, \forall \lambda \in \Omega_\lambda^l, \tag{10}$$

where $\hat{s}(\lambda) = (E^l)^T s(E^l\lambda)$. It is clear that if $s(v)$ is monotone in v then $\hat{s}(\lambda)$ is monotone in λ. Thus the descent method **VIP** can be applied for solving (10). In this case, first, we need to find $P(\lambda)$ and $H(\lambda)$, i.e., given λ, find $\hat{\lambda} \in \Omega_\lambda^l$ such that

$$[\hat{s}(\lambda) + \frac{\hat{B}(\lambda)^T}{\alpha}(\hat{\lambda} - \lambda)]^T E^l(\lambda - \hat{\lambda}) \geq 0, \forall \lambda \in \Omega_\lambda^l, \tag{11}$$

where $\hat{s}(\lambda) = (E^l)^T s(E^l\lambda)$ and $P(\lambda)$ is found if $\hat{B}(\lambda) = (E^l)^T D(E^l\lambda)E^l$ and $H(\lambda)$ is found if $\hat{B}(\lambda) = (E^l)^T B(E^l\lambda)E^l$. Second, we note that $\hat{B}(\lambda)$ is positive (semi)definite as required if $B(v)$ is positive (semi)definite.

5 Computational results

In this section, we apply the descent simplicial decomposition method (DSDM) to the network equilibrium problem. We consider a network $G(N, A)$ where N is a set of nodes including zones where traffic demands are generated and attracted and A is a set of arcs. We define v is a set of feasible arc flows; v^* is a set of equilibrium flows; $s(v) = (s_a(v))$ is a set of arc costs; $I = (i)$ is a set of all pairs; K_i is a set of paths for the OD pair i; $K = (K_i)$ is a set paths for all the OD pairs; $g = (g_i)$ is the set of demands for all the OD pairs; $h = (h_k)$ is the set of path flows; Ω is the feasible region of the arc flows defined as

$$\Omega = \{v | v = \delta h, \sum_{k \in K_i} h_k = g_i, h_k \geq 0, \forall k \in K_i, \forall i \in I\};$$

and δ is the arc-path incidence matrix.

Thus the network equilibrium problem which has been formulated as the variational inequality problem (see Smith, 1979, 1983, Dafermos, 1980, and Bertsekas and Gafni, 1982) is to find $v^* \in \Omega$ such that

$$s(v^*)^T(v - v^*) \geq, \forall v \in \Omega.$$

The adaptation of the DSDM for the network equilibrium problem is simple, where in step 1 and 4, the extreme points are generated by performing all-or-nothing assignments with applications of well known shortest path algorithms.

As we know, the descent type methods have been developed for the monotone variational inequality problem in the framework of Wu et al (1991) which we will discuss together with the VIP algorithm for the solution of the restricted variational inequality problem. In this case, the results will be focussed on two parts. The first part is devoted to the various solution methods for the restricted variational inequality problem and the second part is for the descent simplicial decomposition methods.

The various solution methods for the restricted variational inequality problem at iteration l are defined as follows.

- The optimization based Newton method (OBNTM). This is the **VIP** discussed above $\alpha = 1$ and $\hat{B}(\lambda) = (E^l)^T \nabla s(v) E^l = (E^l)^T \nabla s(E^l \lambda) E^l$.

- The Newton method (NTM). This is the descent Newton method without line search (unit step size) where $\alpha = 1, \hat{B}(\lambda) = (E^l) \nabla s(v) E^l = (E^l) \nabla s(E^l \lambda) E^l$ and $t^l = 1$.

- The optimization based projection method (OBPM). This is the method first proposed by Fukushima (1989) where $\alpha = \alpha_1, \hat{B}(\lambda) = \bar{B}$, $\bar{B}$ is constant symmetric and positive definite matrix and $P(\lambda) = H(\lambda)$.

- The projection method (PM). This is the method proposed by Dafermos (1981) and Bertsekas and Gafni (1982) where $\alpha = \alpha_1, \hat{B}(\lambda) = D$ (D is constant and symmetric positive definite matrix) and $t^l = 1$.

- The optimization based linearized Jacobi method (OBLJM). This is the method developed by Wu et al (1991) where $\hat{B}(\lambda) = (E^l)^T Diag(\nabla s(v)) E^l = (E^l)^T Diag(\nabla s(E^l \lambda)) E^l$ and $P(\lambda) = H(\lambda)$. Note that the OBLJM in Wu et al (1991) requires a stronger condition for convergence where $H(\lambda) = P(\lambda)$.

- The linearized Jacobi method (LJM). This is discussed in Wu et al (1991) where $\alpha = \alpha_1, \hat{B}(\lambda) = (E^l)^T Diag(\nabla s(v)) E^l = (E^l)^T Diag(\nabla s(E^l \lambda)) E^l$ and $t^l = 1$.

	$\hat{B}(\lambda) = \hat{B}$	$\hat{B}(\lambda) = (E^l)^T Diag(\nabla s(v))E^l$	$\hat{B}(\lambda) = (E^l)^T \nabla s(v) E^l$
with line search	OBPM	OBLJM	OBNTM
without line search	PM	LJM	NTM

Table 1: Various methods for variational inequality problem

See Table 1 for a summary of these methods. In the following, we define:

$IT_D.$: the number of the iterations for the restricted variational inequality problem;

$IT_S.$: the number of the simplicial iterations;

$OBJ.$: the relative general gap function, i.e.,
$[-s(v)^T(P(v)-v) - \frac{1}{2\alpha_1}(P(v)-v)^T D(v)(P(v)-v)]/s(v)^T v$;

GAP : the relative gap value, i.e., $-s(v)^T(e^*-v)/s(v)^T v$ where $e^* = \arg\min_{e\in\Omega} s(v)e$;

$DIF.$: the relative solution difference, i.e., $||H(v)-v||_2/||v||_2$

where $v = E^l\lambda$. In the figures, E-n represents 10^{-n}. The various algorithms have been implemented in a high level mathematical programming language GAMS (Brooke et al, 1988). One of the advantages of using the GAMS is the efficiency of implementation of the algorithms under the investigation which can provide us most information in order to understand the performances of the both **VIP** and **DSDM**.

The problem data is given next. The network consists of 6 nodes (including 3 zones) and 14 directed links (see Figure 1). The fixed o-d demand matrix OD

$$OD = \begin{pmatrix} 0 & 20 & 10 \\ 4 & 0 & 15 \\ 7 & 9 & 0 \end{pmatrix}$$

and the cost structure

$$s_{ij} = b^0_{ij} + b^1_{ij}(1 + (v_{ij} + \rho v_{ji})^p)$$

link	b_{ij}^0	b_{ij}^1
(1,4)	1	5
(1,5)	2	2
(2,6)	3	3
(3,6)	1	4
(4,1)	2	3
(4,5)	1	1
(4,6)	1	1
(5,1)	3	5
(5,4)	2	2
(5,6)	1	10
(6,2)	2	2.5
(6,3)	2	2
(6,4)	1	9
(6,5)	3	8

Table 2: Parameters of the cost function

where ρ and p are two positive parameters and b_{ij}^0 and b_{ij}^1 are given in the Table 2. $D(v) = I$ where I is an identity matrix.

5.1 Results on part one

We have conducted many computations on the test problem with various parameters. In the following, we use $\beta = 0.5$ as defined in the **VIP** for the computations. Based on the computational experience, we have the following observations where only two extreme points are included in the unit simplex:

α	α_1	ρ	p
1.3	0.2	0.9	4

Table 3: Parameters of the cost function

1. LJM is better then the PM if a computation of $Diag(\nabla s(v))$ is not expensive in terms of DIF. and OBJ.

2. LJM is in general a good convergent method; the more asymmetric, the $\nabla s(v)$ is, the smaller the α is for a good convergence.

3. NTM converges very fast if it is convergent. There is no need to adjust α which is alway equal to 1. However, it takes more time to solve the asymmetric affine variational inequality problem.

4. The optimization based approachs are easy to convergent (very robust in the sense of selection of α) as compared with the methods without line search. Consider the LJM, the OBLJM and the OBNTM with the following parameters (see Table 3). The results are shown in the Figure 2. Since, in this case, the problem is very asymmetric and highly nonlinear, the LJM is not convergent and the OBLJM is convergent but slowly while the OBNTM has a fast convergence.

α	α_1	ρ	p
0.95	0.2	0.1	2

Table 4: Parameters of the cost function

5. When both ρ ans p are small, the OBLJM behaves in a similar way of OBNTM. See the Table 4 and Figure 3 for the parameters. In this case, $B(\lambda)$ is a good approximation of the $(E^l)^T \nabla s(v) E^l$.

6. The OBLJM is in general recommended in this case. Though the OBNTM converges faster, it requires more computational time for finding $H(\lambda)$.

7. When $\alpha = \alpha_1$ are very small, the **VIP** may encount no descent direction at certain iterations as the theoretical results expected. See a example for the OBLJM where $P(\lambda) = H(\lambda)$ from the Table 5 for parameters and Table 6 for computational results. In this case, the problem is very asymmetric. We found that the OBLJM is not convergent if α_1 and α are fixed at small values, while they are increased from 0.008 (case 1) to 0.2 (case 2) and 2.0 (case 3), we have the convergent results, which is expected in our previous remark. It is interesting to note that in the iterative methods, α should be selected smaller than a certain value in order to have a contraction. (even if α is sufficiently small, the PM is still convergent, while this may not the case for the LJM). From this point of view, the descent methods have a wider

	α	α_1	ρ	p
case 1	0.008	0.008	0.9	2
case 2	0.2	0.2	0.9	2
case 3	2.0	2.0	0.9	2

Table 5: Parameters of the cost function

	case1	case 2	case 3
IT_D.	$OBJ.$	$OBJ.$	$OBJ.$
1	0.002	0.051	0.437
2	0.002	0.035	0.371
3	0.002	0.021	0.013
4	0.002	0.012	0.005
5	0.002	0.006	
6	0.002	0.003	

Table 6: Computational results of changing α_1

rang of selection of α than the iterative methods if α is not adjusted. It is also observed that in case 1 and case 3, the linearized Jacobi method (i.e., OBLJM with stepsize being one) is not convergent, while, in case 2, this method is convergent as the OBLJM. This point will be further studied in a near future.

We mention that theoretically for the OBLJM and the OBPM require that $\alpha_1 = \alpha$ in order to have $P(\lambda) = H(\lambda)$. As observed, even when $\alpha_1 \neq \alpha$, we still obtain convergent results.

α	α_1	ρ	p
1	0.2	0.9	2

Table 7: Parameters of the cost function

5.2 The results on part two

This part of analysis is devoted to the overall behaviour of the simplicial iterations. The following results were conducted by using the DSDM-OBLJM, i.e., using the DSDM where the restricted variational inequality problem is solved by the OBLJM and the DSDM-OBNTM, i.e., using the DSDM where the restricted variational inequality problem is solved by the OBNTM for the same number of simplicial iterations. See the Table 7 and Figure 4. The observations are that the DSDM-OBNTM outperforms the DSDM-OBLJM since the restricted variational inequality problem can not be solved exactly for the same number of iterations and the OBNTM provides a more accurate solution.

6 Conclusions

In this paper, we have developed a family of the descent methods for the monotone (not strongly) variational inequality problem where the proof of the convergence is given. According to these results a family of flexible solution algorithms called descent simplicial decomposition method (DSDM) was established with a discussion of various steps.

Computational results are reported for the algorithms as compared with some recent developed methods. The preliminary results shows the DSDM is promising.

We mention that the algorithms developed for the restricted variational inequality problem take into account a very special constraint structure, i.e., unit-simplex where the solutions of both asymmetric and symmetric affine variational inequality problems are very efficient. However if in general such a solution of affine variational inequality problem may be every expensive and in this case the Marcotte and Dussault (1989)'s method should be recommended where the solution for descent direction involves only a linear program in a primal and dual space.

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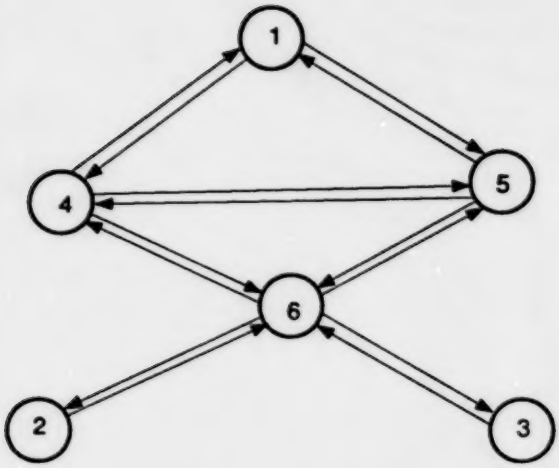

Figure 1: The six zone network

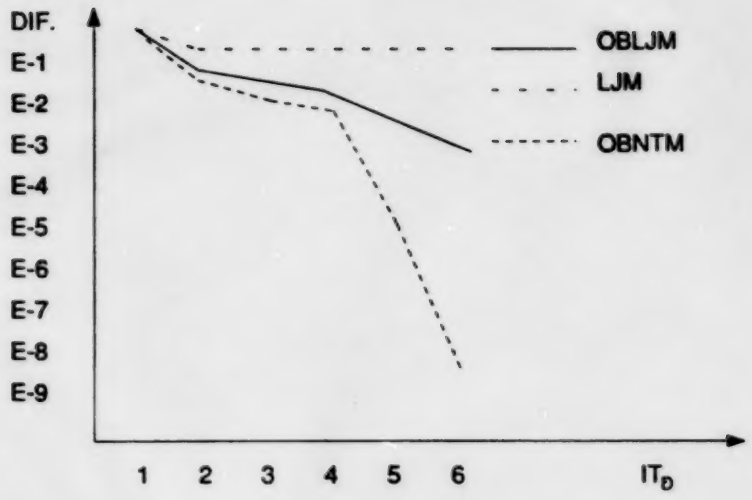

Figure 2: Computational results of OBLJM, LJM and OBNTM

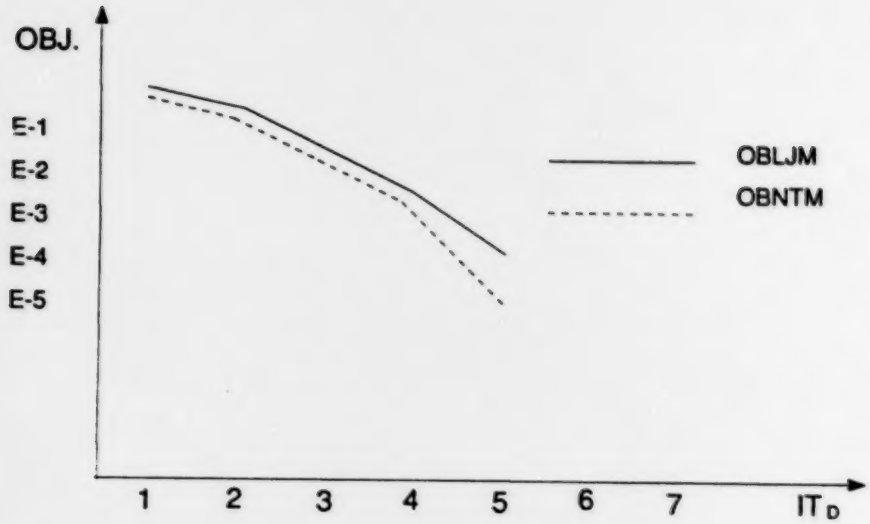

Figure 3: Computational results on OBLJM and OBNTM

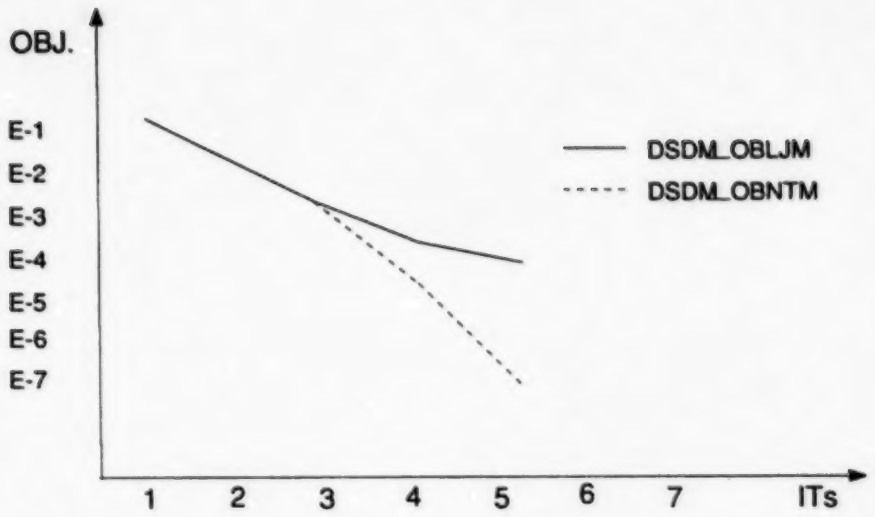

Figure 4: Computational results on DSDM-OBLJM and DSDM-OBNTM